Math 210C Lecture 5 Notes

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1 Hilbert's Nullstellensatz and Spectra of Rings

1.1 Hilbert's Nullstellensatz

Last time, we were proving Hilbert's Nullstellensatz. Let $R = K[x_1, \dots, x_n]$, where K is algebraically closed.

Theorem 1.1 (Hilbert's Nullstellensatz). *I and V provide mutually inverse, inclusion reversing bijections $\{$radical ideals of $K[x_1, \dots, x_n]\} \leftrightarrow \{$algebraic sets in $\mathbb{A}^n_K\}$.*

Proof. It remains to prove that $I(V(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of R. Since R is noetherian, $\mathfrak{a} = (g_1, \dots, g_k$, so $R[y] = K[x_1, \dots, x_n, y] \supseteq J = (g_1, \dots, g_j, 1 - f_y)$. If $a \in V(J)$, where $a = (a_1, \dots, a_{n+1}$, then $b := (a_1, \dots, a_n) \in V(\mathfrak{a})$, so $f(b) = 0$. Then $(1 - fy)(a) = 1 - f(b)a = 0$, which is impossible. So $V(J) = \varnothing$, which means that $J = R[y]$.

Let $I = h(1 - fy) + \sum_{i=1}^k h_i g_i$, where $h, h_i \in R[y]$. Let $N = \max(\{\deg_y(h_i) : 1 \leq i \leq k\} \cup \{h_y\})$. If $z = y^{-1}$, then $z^{N+1} = h'(z - f) + \sum_{i=1}^k h_i' g_i$, where $h', h_i' \in R[z]$. Then setting $z = f$ gives $f^{N+1} = \sum_{i=1}^k h_i'(f) g_i \in \mathfrak{a}$, so $f \in \sqrt{\mathfrak{a}}$. $\square$

Remark 1.1. If $S \subseteq R$, then $V(S) = V(\sqrt{(S)})$, and $I(V(S)) = \sqrt{(S)}$. If $Z \subseteq \mathbb{A}^n_K$, then $I(Z) = I(\overline{Z})$, and $V(I(Z)) = \overline{Z}$, where $\overline{Z}$ is the closure of Z in the Zariski topology.

Definition 1.1. An algebraic set is **irreducible** if it is not a union of any two proper algebraic subsets.

Corollary 1.1. *I and V restrict to mutually inverse bijections $\{$prime ideals of $R\} \leftrightarrow \{$irreducible algebraic sets in $\mathbb{A}^n_K\}$.*

Proof. If Z is an algebraic set, then $Z = V(I)$, where I is a radical ideal. Then $I = \bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is primary. Then $I = \sqrt{I} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i}$, so $\mathfrak{q}_i = \sqrt{\mathfrak{q}_i}$ is prime for all i. Z is irreducible if and only if I is irreducible, which means that I is prime (since it it radical and $n = 1$). $\square$

1.2 Spectra of rings

Definition 1.2. The **spectrum** $\operatorname{Spec}(R)$ of a ring R is its set of prime ideals.

Example 1.1. If R is a PID, then $\operatorname{Spec}(R) = \{(0)\} \cup \{(f) : f \text{ is irreducible}\}$.

Example 1.2. In $R = K[x_1, \dots, x_n]$, $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_n)$. The corresponding algebraic subsets decrease in dimension as se go up the chain.

We can define V and I (slightly differently) for general rings.

Definition 1.3. If $T \subseteq R$, then $V(T) = \{\mathfrak{p} \in \operatorname{Spec}(R) : T \subseteq \mathfrak{p}\}$. If $Y \subseteq \operatorname{Spec}(R)$, then $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$.

Comparison with V and I for $R = K[x_1, \dots, x_n]$:

- If $T \subseteq R$ and $\mathfrak{m} \subseteq R$ is maximal, then $T \subseteq \mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$. So $(a_1, \dots, a_n)$ is in the vanishing locus of T. If we think of this point as $\mathfrak{m}$, then $\mathfrak{m} \in V(T)$. For our current definition of V, $\mathfrak{p} \in V(T) \iff T \subseteq \mathfrak{p}$. So V extends the notion of the vanishing locus.

- T corresponds to $V \subseteq \mathbb{A}^n_K$. Then $T \subseteq \mathfrak{p} \iff V$ contains the irreducible algebraic set corresponding to $\mathfrak{p}$. This is iff V contains all points corresponding to maximal ideals $\supseteq \mathfrak{p}$. On the other hand, $T \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \in V(T)$. Then $V(T)$ contains all maximal ideals containing $\mathfrak{p}$.

- If $Z \subseteq \mathbb{A}^n_K$, then the radical ideal I corresponding to Z has the form $I = \bigcap_{i=1}^n \mathfrak{p}_i$ for $\mathfrak{p}_i$ prime. If we define $Y \subseteq \operatorname{Spec}(R)$ to be $Y = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, then $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} = \bigcap_{i=1}^n \mathfrak{p}_i = I$.

This should give you a rough sense that these are natural extensions of our definitions to the set of all prime ideals.

Note that $V(T) = V((T))$. If I is an ideal, $V(I) = V(\sqrt{I})$ because if $I \subseteq \mathfrak{p}$, then $\sqrt{I} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$.

Lemma 1.1. *$V((0)) = V(0) - \operatorname{Spec}(R)$, and $V(R) = \varnothing$.*

1. *If $\mathfrak{a}, \mathfrak{b}$ are ideals of R, then $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.*

2. *If $\{\mathfrak{a}_j : j \in X\}$ is a set of ideal, then $\bigcap_{j \in X} V(\mathfrak{a}_j) = V(\bigcup_{j \in X} \mathfrak{a}_j) = V(\sum_{j \in X} \mathfrak{a}_j)$.*

Definition 1.4. The **Zariski topology** on $\operatorname{Spec}(R)$ is the unique topology with closed sets the $V(I)$, where $I \subseteq R$ is an ideal.

Remark 1.2. In $\operatorname{Spec}(R)$, if $\mathfrak{m}$ is maximal, then $\{\mathfrak{m}\}$ is closed. These are the only closed points, so we call maximal ideals **closed points** of R. If $\mathfrak{p}$ is prime but not maximal, then there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{p} \subsetneq \mathfrak{m}$. Then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) \supseteq \{\mathfrak{p}, \mathfrak{m}\}$. So $\{\mathfrak{p}\}$ is closed if and only if $\mathfrak{p}$ is maximal.